is $a_k = -1.867$, whereas numerical integration[3] gives $a_k = -1.913$.

Finally, for the shear flow over a flat plate ($U^{(1)} = 1$, $U^{(2)} = 0$, $k = 0$) the foregoing method gives $a_\Omega = -3.021$. The corresponding value obtained in Ref. 4 is -3.126.

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Zonal Flow Inside an Impulsively Started Rotating Sphere

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Introduction

THE concern here is with the manner in which the rigid-body rotating flow field inside a hollow sphere is established when the motion starts impulsively from rest. Mathematically, the problem is also equivalent to the decay of an initial rigid-body rotating flow when the motion of the sphere is suddenly arrested. We consider, specifically, a hollow sphere of radius a, filled with a viscous, incompressible fluid which is initially at rest. At time $t = 0$, the sphere is started impulsively into steady rotation about a diameter with angular velocity Ω = const. Because of the viscous no-slip boundary condition, a boundary layer forms and diffuses inwardly. Eventually, the fluid rotates as a rigid body.

The zonal or circumferential component of flow w will drive a secondary flow (with radial velocity u and meridional velocity v) through the action of centrifugal forces on the fluid. However, during the early (and also the late) stages of the flow, that is, for small (and large) values of Ωt, the secondary flow will be small relative to the primary zonal flow ($u \ll w, v \ll w$). Consequently, w can be found approximately

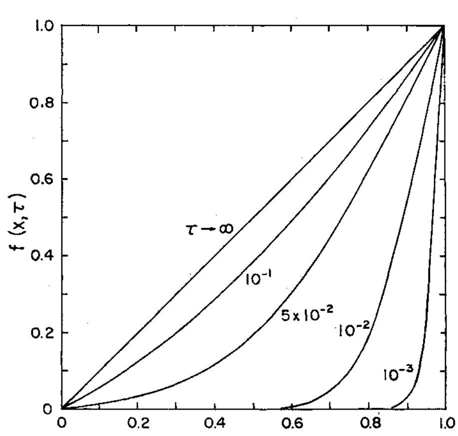

Fig. 1 Nondimensional zonal velocity field as a function of nondimensional radius and time.

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from a linearized Navier-Stokes equation in which the influence of the secondary flow on the primary flow is neglected. This linear equation is solved exactly for $w(r, \theta, t)$. The resulting solution is of boundary-layer type; it predicts that rigid-body motion of the fluid is approached in a time of order a^2/ν; the viscous torque at the surface is found as a function of time.

Formulation

The continuity equation and the Navier-Stokes equations for the flow of a viscous, incompressible fluid are

$$\nabla\cdot\mathbf{v} = 0 \qquad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right) + (\nabla\times\mathbf{v})\;X\mathbf{v} = -\frac{1}{\rho}\nabla p - \nu\nabla X(\nabla X\mathbf{v}) \qquad (2)$$

Throughout, we use nonrotating spherical polar coordinates r, θ, ϕ (with θ the colatitude) and velocity components u, v, w in the directions of increasing r, θ, ϕ, respectively. Thus, w is the zonal flow in planes perpendicular to the axis of rotation, and u, v constitute the secondary flow in planes containing the axis of rotation. The ϕ component of Eq. (2) with $\partial/\partial\phi \equiv 0$, because of rotational symmetry, is

$$w_t + uw_r + r^{-1}(vw_\theta + uw + vw\cot\theta) = \nu r^{-2}[(r^2w_r)_r + (\sin\theta)^{-1}(w_\theta\sin\theta)_\theta - (\sin\theta)^{-2}w] \qquad (3)$$

where subscripts indicate partial differentiation. Consideration of the initial and boundary conditions as well as the steady-state rigid-body rotation leads to the conclusion that the nonlinear terms in Eq. (3) are initially and ultimately zero throughout the flow and are always zero at the boundary and the axis of rotation. Therefore, a reasonable approximation is to omit them in determining w. Another way of saying this is that, for small values of Ωt, the boundary layer will be thin relative to the sphere's radius; consequently, the viscous terms will be large, as will the nonsteady acceleration, whereas the convective acceleration retains its normal values. Without further justification here, we restrict attention to the linearized form of Eq. (3) which can be written nondimensionally as

$$f_\tau = f_{xx} + 2x^{-1}f_x - 2x^{-2}f$$

where

$$x = r/a \qquad \tau = \nu t/a^2$$
$$w(r, \theta, t) = a\Omega\sin\theta f(x, \tau) \qquad (4)$$

It is worth noting that the linear equation requires only a $\sin\theta$ dependence for w. The nondimensional time τ has two interpretations. Since the boundary-layer thickness δ is expected to be of order $(\nu t)^{1/2}$, τ is essentially the ratio of δ^2 to a^2. Or, in terms of the Reynolds or Taylor number for this flow, $R = a^2\Omega/\nu$, τ is the ratio of the sphere's angular rotation Ωt to the Taylor number. Since the linearization is expected to be valid for small τ, the solution is believed to be a good approximation for angles of rotation small compared to the Taylor number.

The boundary and initial conditions on $f(x, \tau)$ are

$$f(1, \tau) = 1 \qquad f(0, \tau) = f(x, 0) = 0 \qquad (5)$$

Solution

The parabolic equation (4) is, of course, linear but has variable coefficients. It is similar to the standard diffusion equation for heat conduction, but the analogy is not exact because the viscous terms in the Navier-Stokes equation are not given by the Laplacian of the velocity except in cartesian coordinates.

The boundary-value problem posed by Eqs. (4) and (5) is easily solved by introducing the Laplace transform with respect to τ. The transformed problem has an exact solution in terms of spherical Bessel functions. The inversion requires some care but is straight forward. The result is

$$f(x, \tau) = x - 2x^{-2} \sum_{n=1}^{\infty} \zeta_n^{-3}(\cos\zeta_n)^{-1}(x\zeta_n\cos x\zeta_n - \sin x\zeta_n)e^{-\zeta_n^2\tau} \quad (6)$$

where ζ_n is the nth positive root of the transcendental equation

$$\zeta = \tan\zeta \quad (7)$$

The first sixteen values of ζ_n are tabulated in Jahnke and Emde.[1] An analytic approximation for ζ_n accurate to 0.1% is given by Kestin and Persen[2] as

$$\zeta_n \doteq \frac{2n+1}{4}\pi + \left[\left(\frac{2n+1}{4}\right)^2 \pi^2 - 1\right]^{1/2} \quad (8)$$

Discussion and Results

The first term in Eq. (6) is the steady-state solution (rigid-body rotation), and the summation gives the transient. The reason that the linearized equation (4) gives the correct steady-state solution is, of course, that the neglected terms vanish as $\tau \to \infty$. Equation (6) has been calculated numerically, and the results are shown in Fig. 1. At nondimensional time $\tau = 0.1$, $f(x, \tau)$ has not yet approached closely to the asymptotic value. However, when $\tau = 1$, the value of $f(x, \tau)$ agrees with the steady-state solution to at least five significant digits. Therefore, as was easily surmised by Stewartson and Roberts,[3] the time required for the establishment of rigid-body rotation of the fluid is of order a^2/ν.

It is worth remarking that, even though the equation solved is linear, with the nonlinear terms completely omitted, the solution is still of boundary-layer type in the sense that for small ν, i.e., small τ, the viscous effects are confined to a narrow region adjacent to the wall. This corresponds to the fact that boundary-layer behavior is not so much dependent upon nonlinearity in the differential equations as it is on the smallness of the coefficient that multiplies the most highly differentiated terms (see, for example, Carrier[4]). If δ is defined as the radial distance from the surface at which the circumferential flow has fallen to 1% of its value at the wall (on the same radius), then, for values of $\tau \leq 0.01$,

$$\delta \doteq 3.7a\tau^{1/2} = 3.7(\nu t)^{1/2} \quad (9)$$

Although the present analysis neglects the effect of the secondary flow on the zonal flow field, the viscous torque at the sphere is still expected to be obtained fairly accurately. This follows because the shear stress at $r = a$ depends only on the radial gradient of (w/r) evaluated right at the surface, and in that region the secondary flow is always small since it vanishes right at $r = a$. Integrating the product of shear stress and moment arm over the surface of the sphere, we find that the torque is

$$T = \int_0^{2\pi}\int_0^{\pi} \mu a^4 \left[\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right]\Bigg|_{r=a} \sin^2\theta \, d\theta d\phi = \frac{16\pi}{3}\mu a^3\Omega \sum_{n=1}^{\infty} e^{-\zeta_n^2 n\tau} \quad (10)$$

For large values of τ, the first term in the summation dominates, and the torque decays exponentially to zero. As $\tau \to 0$, the torque becomes infinite since the motion was started impulsively from rest. The asymptotic expansion of $T(\tau)$ for small τ is†

$$T(\tau) \sim [8(\pi)^{1/2}/3]\mu a^3\Omega\tau^{-1/2} = [8(\pi)^{1/2}/3](\rho\mu)^{1/2}a^4\Omega t^{-1/2} \quad (11)$$

† The author is indebted to F. Ursell, University of Manchester, for pointing this out.

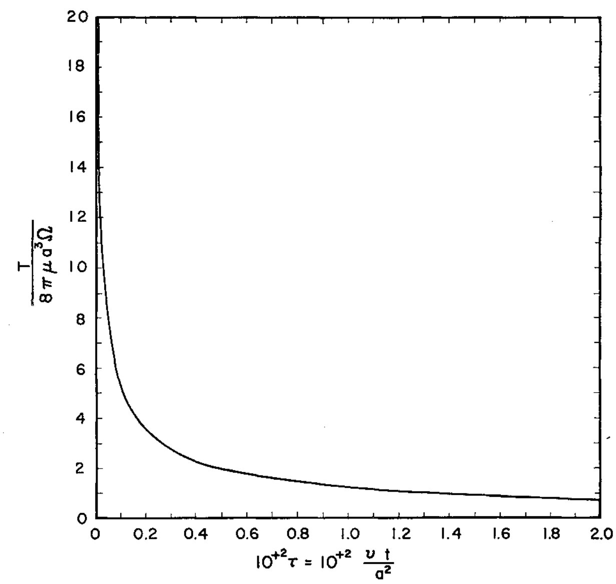

Fig. 2 Nondimensional viscous torque as a function of nondimensional time.

The torque as calculated from Eq. (10) is plotted as a function of time in Fig. 2. It is notable that, in the later stages of flow, the torque, while still significantly different from its asymptotic value of zero, approaches that value very slowly.

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A Property of Cotangential Elliptical Transfer Orbits

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I. Introduction

COTANGENTIAL transfer between two planar, confocal, noncoaxial ellipses has been investigated by many authors. The purpose of this paper is to prove that generally this type of transfer does not yield a minimum energy transfer. Also it will be shown that between two coplanar, confocal, noncoaxial ellipses, there exist two cotangential transfer orbits that yield a relative optimum transfer. One of these two orbits, in the special case of two coaxial terminal ellipses, turns out to be the well-known coapsidal, absolute optimum two-impulses transfer. A practical application of this property is in the case where one or both of the terminal ellipses are almost circular.

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